

FIBER DETECTION FOR STATE SURFACES

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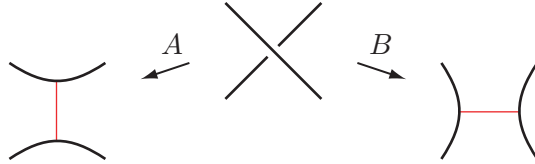
ABSTRACT. Every Kauffman state σ of a link diagram $D(K)$ naturally defines a state surface S_σ whose boundary is K . For a *homogeneous* state σ , we show that K is a fibered link with fiber surface S_σ if and only if an associated graph \mathbb{G}'_σ is a tree. As a corollary, it follows that for an adequate knot or link, the second and next-to-last coefficients of the Jones polynomial are obstructions to certain state surfaces being fibers for K .

This generalizes a theorem from [4], with a dramatically simpler proof.

1. INTRODUCTION

In the 1930s, Seifert gave an algorithm that starts with a link diagram $D(K)$ and produces an orientable surface whose boundary is K [10]. The algorithm works as follows. For every crossing of D , smooth the diagram near the crossing by following an orientation on K . This gives a disjoint union of circles in the projection plane. These circles bound a number of disks, disjointly embedded in the ball below the projection plane. Then, these disks can be joined by half-twisted bands at the crossings to give a surface $S \subset S^3$, such that $\partial S = K$.

Seifert's construction has a natural generalization. At each crossing, there are two possible smoothings, or *resolutions* of the crossing, as depicted in Figure 1. A *Kauffman state* is a choice of A - or B -resolution at each crossing. As in Seifert's construction, a state σ gives rise to a union of circles in the projection plane. These circles bound disjoint disks, which can be joined by half-twisted bands to give a *state surface* S_σ . See Figure 2.

FIGURE 1. A - and B -resolutions at a crossing of D .

Another common example of a state surface is the two checkerboard surfaces of an alternating diagram. The regions in the complement of $D(K)$ can be checkerboard colored, black and white. If $D(K)$ is alternating and σ is the all- A state, we obtain a collection of circles tracing the boundaries of the black regions. Joining these disks by half-twisted bands gives the (black) all- A checkerboard surface S_A . Similarly, the all- B state of an alternating diagram gives the checkerboard surface S_B , containing all the white regions.

The goal of this paper is to characterize when a state surface S_σ is a fiber in a fibration of $S^3 \setminus K$ over S^1 . For checkerboard surfaces, the strikingly simple answer is that S_A is a fiber if and only if D is a connected sum of positive 2-braids. (See Lemma 4 below.) Stating our result in general requires a handful of definitions.

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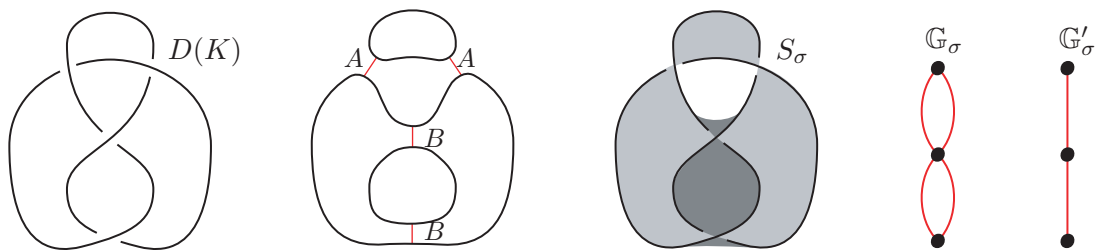


FIGURE 2. Left to right: a diagram $D(K)$. A homogeneous state σ , coming from Seifert's algorithm. The state surface S_σ corresponding to σ . The graph \mathbb{G}_σ embeds into S_σ as a spine. The reduced graph \mathbb{G}'_σ .

Given a state σ , every crossing of D belongs to a region of bounded by state circles. The state σ is called *homogeneous* if all crossings in the same region carry the same (A or B) resolution. For example, the state shown in Figure 2 is homogeneous. This notion was introduced by Cromwell [2] for the Seifert state, and easily extends to other states.

The choices that lead to a Kauffman state σ can be conveniently encoded in a *state graph* \mathbb{G}_σ . This graph has one vertex for each state circle of σ . Each crossing x of D gives rise to an edge between the state circles at the resolution of x . (In Figures 1 and 2, these edges are shown in red, lighter than the link projection.) From the graph \mathbb{G}_σ , we construct a reduced graph \mathbb{G}'_σ by removing all duplicate edges between a pair of vertices. Our main result is that this reduced graph carries fibering information.

Theorem 1. *Let σ be a homogeneous state of a link diagram $D(K)$. Then $S^3 \setminus K$ fibers over the circle with fiber S_σ if and only if the reduced graph \mathbb{G}'_σ is a tree.*

The topology of state surfaces was recently studied by Ozawa [9]. For homogeneous states, he showed that the surface S_σ is essential in $S^3 \setminus K$ if and only if the state σ is *adequate*, meaning that \mathbb{G}_σ has no 1-edge loops (equivalently, \mathbb{G}'_σ has no 1-edge loops). Since a tree has no loops of any length, all the states where S_σ is a fiber must be adequate.

Theorem 1 generalizes a recent result of the author with Kalfagianni and Purcell [4, Theorem 5.11]. That theorem says that the all- A state surface S_A is a fiber in $S^3 \setminus K$ if and only if \mathbb{G}'_A is a tree (and similarly for S_B). The all- A and all- B graphs $\mathbb{G}'_A, \mathbb{G}'_B$ are particularly worthy of attention due to their connection to the Jones polynomial of K . This is a Laurent polynomial invariant of K , which can be written in the form

$$J_K(t) = \alpha t^k + \beta t^{k-1} + \dots + \beta' t^{m+1} + \alpha' t^m,$$

so that the second and next-to-last coefficients of $J_K(t)$ are β and β' , respectively. Stoimenow [12] and Dasbach and Lin [3] have observed that if D is A -adequate (meaning the all- A state is adequate), then $|\beta'| = 1 - \chi(\mathbb{G}'_A)$. Similarly, if D is B -adequate, then $|\beta| = 1 - \chi(\mathbb{G}'_B)$. As a result, the following is an immediate corollary of either Theorem 1 or [4, Theorem 5.11].

Corollary 2. *Let K be a link that admits a connected, A -adequate diagram D . Then $S^3 \setminus K$ fibers over S^1 with fiber the state surface $S_A = S_A(D)$ if and only if $\beta' = 0$.*

In other words, the next-to-last Jones coefficient β' is precisely the obstruction to $S^3 \setminus K$ being fibered in with fiber surface S_A . Similarly, for B -adequate links, the second coefficient β is the obstruction to $S^3 \setminus K$ being fibered with fiber surface S_B . This corollary is one of the first effective connections between quantum knot invariants and classical geometric topology.

In summary, Theorem 1 improves on [4, Theorem 5.11] in two different ways. First, Theorem 1 generalizes the fibering criterion from the all- A and all- B states to all homogeneous states. It is an interesting open problem to see whether the graphs \mathbb{G}_σ and \mathbb{G}'_σ of a homogeneous state σ are somehow reflected in the Jones polynomial or one of its relatives. If such a connection exists, then Corollary 2 will also generalize.

Second, the proof of Theorem 1 is dramatically shorter and simpler. The proof of [4, Theorem 5.11] involves the detailed study of a polyhedral decomposition of $S^3 \setminus S_A$, and much work is expended to show that the polyhedra have desirable properties [4, Chapters 2–4]. By contrast, the proof of Theorem 1 contained in this paper is short and straightforward. As a consequence, one also obtains a short and readily digestible proof of Corollary 2.

Our proof builds up the surface S_σ inductively via Murasugi sums (see Figure 5), applying results of Gabai [5] to deduce fibering information. This inductive approach is very similar in spirit to the methods used by Ozawa [9]. It is also fruitfully exploited in a recent preprint of Girão to prove a fibering criterion for augmented links [6].

2. PROOF

Before beginning the main proof of Theorem 1, we wish to dismiss a few special cases. If $D(K)$ depicts an unknot with no crossings, then \mathbb{G}'_σ is a single vertex, and the spanning disk S_σ is a fiber of the solid torus $S^3 \setminus K$. Thus the theorem holds trivially. If $D(K)$ is a split diagram, then \mathbb{G}'_σ is a disconnected graph (which cannot be a tree), and $S^3 \setminus K$ is reducible (hence cannot be fibered). Again, the theorem holds trivially in this case.

If the state σ is not adequate, i.e. \mathbb{G}_σ has a 1-edge loop, the reduced graph \mathbb{G}'_σ cannot be a tree. In this setting, Ozawa has observed that the 1-edge loop in \mathbb{G}_σ gives rise to a boundary compression disk for S_σ , hence the surface cannot be a fiber [9, Remark on page 6]. Thus the theorem holds for non-adequate states.

For the remainder of the paper, we work under the assumptions that $D(K)$ is connected and has at least one crossing, and that the state σ is adequate. With these simplifying assumptions, the proof of Theorem 1 proceeds by induction on the number of cut vertices in the graph \mathbb{G}'_σ . Recall that a *cut vertex* is a vertex that separates \mathbb{G}'_σ .

The base case of the induction involves prime, alternating diagrams. Recall that a link diagram $D(K)$ is called *prime* if, for every simple closed curve γ in the projection plane that intersects $D(K)$ transversely in two points, one of the two sides of γ contains no crossings. In other words, $D(K)$ fails to be prime precisely when it is the connected sum of two non-trivial diagrams.

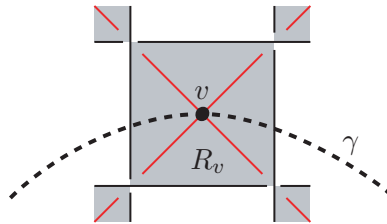


FIGURE 3. If $D(K)$ is an alternating diagram, and v is a cut vertex of the checkerboard graph \mathbb{G}_A , then the corresponding black region R_v separates the diagram as a connected sum (with summands on opposite sides of γ).

Lemma 3. *Let σ be a homogeneous state of $D(K)$. Then \mathbb{G}'_σ does not contain any cut vertices if and only if all of the following hold: $D(K)$ is prime and alternating, and σ is the all- A or all- B state.*

Proof. For the “if” direction, suppose that $D(K)$ is alternating, and σ is the all- A or all- B state. Without loss of generality, σ is the all- A state. Then \mathbb{G}_σ is the all- A checkerboard graph of D . The graph $\mathbb{G}_\sigma = \mathbb{G}_A$ naturally embeds as a spine of the (black) checkerboard surface $S_\sigma = S_A$, with one vertex in each black region of $D(K)$ and one edge running through each half-twisted band.

Suppose, for a contradiction, that $v \in \mathbb{G}'_A$ is a cut vertex. Then v also separates the unreduced graph \mathbb{G}_A . Since \mathbb{G}_A is embedded as a spine of the checkerboard surface S_A , the black region R_v corresponding to v must separate the checkerboard surface, hence also separate the diagram $D(K)$. In other words, there is a simple closed curve γ in the projection plane such that the only black region met by γ is R_v , and such that each component of $\mathbb{R}^2 \setminus \gamma$ contains at least one crossing (these crossings correspond to edges of $\mathbb{G}_A \setminus v$). This curve decomposes $D(K)$ as a connected sum, violating the hypothesis of primeness. See Figure 3.

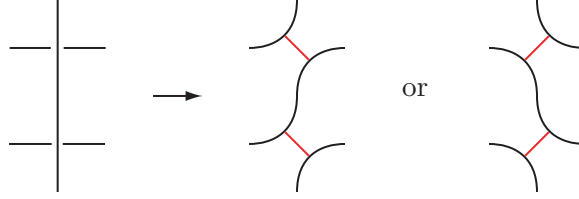


FIGURE 4. If $D(K)$ is a non-alternating diagram, a homogeneous resolution of an over-over strand of D must result in a state circle that has other circles on both sides. The same is true of an under-under strand.

For the “only if” direction, suppose that $D(K)$ is not alternating. Since σ is homogeneous, some state circle of σ (corresponding to a non-alternating segment of D) must have other state circles on both sides. See Figure 4. The corresponding vertex of \mathbb{G}_σ will separate \mathbb{G}_σ , and also \mathbb{G}'_σ . Thus, if \mathbb{G}'_σ contains no cut vertices, $D(K)$ must be alternating.

If D is alternating but σ is not the all- A or all- B state, then some state circle fails to follow the boundary of a region of D . Just as in Figure 4, this implies that this state circle has other circles on both sides, hence \mathbb{G}'_σ has a cut vertex.

Finally, if D is not prime, then there is a simple closed curve γ that meets only one white region and only one black region, with crossings on both sides of γ . One of these regions corresponds to a vertex of \mathbb{G}_σ (depending on whether σ is the all- A or all- B state), and this vertex separates \mathbb{G}'_σ . Therefore, if \mathbb{G}'_σ has no cut vertices, $D(K)$ must be prime and alternating, with σ the all- A or all- B state. \square

The base case of the induction is the following lemma.

Lemma 4. *Suppose $D(K)$ is a prime, alternating diagram with at least one crossing. Then the following are equivalent for the all- A state:*

- (1) \mathbb{G}'_A is a tree.
- (2) \mathbb{G}_A has exactly two vertices.
- (3) $D(K)$ is a positive 2-braid.
- (4) The checkerboard surface S_A is a fiber in $S^3 \setminus K$.

The same equivalence holds for S_B , \mathbb{G}'_B , and negative 2-braids.

Proof. Suppose \mathbb{G}'_A is a tree. If it has only one vertex, then there are no edges, contradicting the hypothesis that $D(K)$ has crossings. If it has three or more vertices, some vertex will be separating, contradicting Lemma 3. Thus \mathbb{G}_A has exactly two vertices, giving (1) \Rightarrow (2).

If \mathbb{G}_A has two vertices, then $D(K)$ has exactly two black regions, connected to each other at each crossing. This is the diagram of a positive 2-braid. Conversely, if $D(K)$ is a positive 2-braid, then \mathbb{G}'_A is a stick with two vertices, which is a tree. Thus (1) \Leftrightarrow (2) \Leftrightarrow (3).

(3) \Rightarrow (4) is a special case of Stallings' theorem [11]: if D is a positive braid, then S_A is a fiber. To see this implication directly, recall Menasco's decomposition of $S^3 \setminus K$ into two ideal polyhedra [7]. The 1-skeleton of each polyhedron is isomorphic (as a planar graph) to the projection graph of $D(K)$. In particular, the faces of the polyhedra can be 2-colored: the union of all the black faces is the checkerboard surface S_A , while the union of all the white faces is the checkerboard surface S_B . In particular, the manifold $S^3 \setminus S_A$ can be obtained by gluing the two polyhedra along white faces only.

If D is a positive 2-braid, then every white face is a bigon. In other words, every polyhedron is combinatorially a prism $P \times I$, where P is an ideal polygon, and the lateral faces are the white bigons. The product structure of $P \times I$ extends as we glue the two polyhedra along their lateral faces, implying that $S^3 \setminus S_A \cong S_A \times I$, hence S_A is a fiber.

For (4) \Rightarrow (3), recall that if D is alternating, prime, and not a 2-braid, then $S^3 \setminus K$ admits a hyperbolic structure [8]. Then, Adams shows that the checkerboard surface S_A is not a fiber [1, Theorem 1.9].

For a more direct proof that (4) \Rightarrow (3), suppose that $S^3 \setminus S_A \cong S_A \times I$. Then, [4, Lemma 4.18] shows that Menasco's polyhedral decomposition must "see" this product structure: every white face must be a product of an ideal edge with I , i.e., an ideal bigon. (Note that when the proof of that lemma is applied to Menasco's well-known polyhedral decomposition, it becomes self-contained, and does not require the machinery developed in [4].) If every white B -region of $D(K)$ is a bigon, these bigons must be joined end to end, implying that $D(K)$ is a positive 2-braid. \square

We are now ready to complete the proof of the main theorem.

Proof of Theorem 1. We proceed by induction on n , where n is the number of cut vertices in \mathbb{G}'_σ . For the base case, let $n = 0$, and recall the running assumption that $D(K)$ has at least one crossing. Then Lemma 3 says that the diagram $D(K)$ is prime and alternating, and σ is the all- A or all- B state. By Lemma 4, S_σ is a fiber if and only if \mathbb{G}'_σ is a tree, as desired.

For the inductive step, suppose $n > 0$ and v is a cut vertex of \mathbb{G}'_σ . Then $\mathbb{G}'_\sigma = \mathbb{G}'_1 \cup_v \mathbb{G}'_2$, where \mathbb{G}'_1 and \mathbb{G}'_2 are subgraphs that are disjoint except at v . The unreduced graph \mathbb{G}_σ , which has the same adjacency relations as \mathbb{G}'_σ , also decomposes as a union of subgraphs \mathbb{G}_1

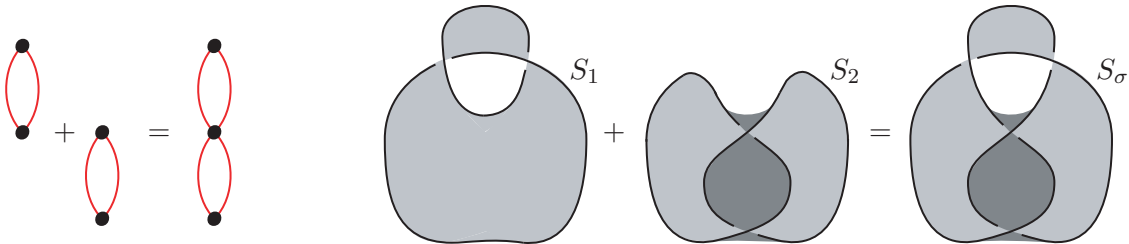


FIGURE 5. Left: The graph \mathbb{G}_σ decomposes as a union of subgraphs \mathbb{G}_1 and \mathbb{G}_2 , joined along the cut vertex v . Right: The corresponding decomposition of S_σ as a Murasugi sum of state surfaces S_1 and S_2 .

and \mathbb{G}_2 that are disjoint except at v . See Figure 5, left. On the diagrammatic side, $D(K)$ decomposes as the Murasugi sum of diagrams $D(K_1)$ and $D(K_2)$, and the state surface S_σ decomposes as the Murasugi sum of state surfaces S_1 and S_2 . See Figure 5.

Note that the Kauffman state σ_i that gives rise to S_i is obtained by restricting σ to the crossings of $D(K_i)$. Thus each S_i is the state surface of a homogeneous state, with reduced graph \mathbb{G}'_i . Note as well that each \mathbb{G}'_i has fewer than n cut vertices. Thus, by the inductive hypothesis, S_i is a fiber in $S^3 \setminus K_i$ if and only if \mathbb{G}'_i is a tree.

Now, we recall Gabai's theorem [5]: S_σ is a fiber if and only if each S_i is a fiber. Clearly, \mathbb{G}'_σ is a tree if and only if each \mathbb{G}'_i is a tree. This completes the proof. \square

REFERENCES

- [1] Colin C. Adams, *Noncompact Fuchsian and quasi-Fuchsian surfaces in hyperbolic 3-manifolds*, *Alebr. Geom. Topol.* **7** (2007), 565–582.
- [2] Peter R. Cromwell, *Homogeneous links*, *J. London Math. Soc.* (2) **39** (1989), no. 3, 535–552.
- [3] Oliver T. Dasbach and Xiao-Song Lin, *On the head and the tail of the colored Jones polynomial*, *Compositio Math.* **142** (2006), no. 5, 1332–1342.
- [4] David Futer, Efstratia Kalfagianni, and Jessica S. Purcell, *Guts of surfaces and the colored Jones polynomial*, arXiv:1108.3370.
- [5] David Gabai, *The Murasugi sum is a natural geometric operation*, *Low-dimensional topology* (San Francisco, CA, 1981), *Contemp. Math.*, vol. 20, Amer. Math. Soc., Providence, RI, 1983, pp. 131–143.
- [6] Darlan Girão, *On the fibration of augmented link complements*, 2011, arXiv:1109.3084.
- [7] William Menasco, *Polyhedra representation of link complements*, *Low-dimensional topology* (San Francisco, Calif., 1981), *Contemp. Math.*, vol. 20, Amer. Math. Soc., Providence, RI, 1983, pp. 305–325.
- [8] ———, *Closed incompressible surfaces in alternating knot and link complements*, *Topology* **23** (1984), no. 1, 37–44.
- [9] Makoto Ozawa, *Essential state surfaces for knots and links*, *J. Aust. Math. Soc.* (to appear), arXiv:math/0609166.
- [10] Herbert Seifert, *Über das Geschlecht von Knoten*, *Math. Ann.* **110** (1935), no. 1, 571–592.
- [11] John R. Stallings, *Constructions of fibred knots and links*, *Algebraic and geometric topology* (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, *Proc. Sympos. Pure Math.*, XXXII, Amer. Math. Soc., Providence, R.I., 1978, pp. 55–60.
- [12] Alexander Stoimenow, *Coefficients and non-triviality of the Jones polynomial*, *J. Reine Angew. Math.* (2011), DOI: 10.1515/CRELLE.2011.047.

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